

# Short and Long Range Screening of Optical Singularities

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Screening of topological charges (singularities) is discussed for paraxial optical fields with short and with long range correlations. For short range screening the charge variance  $\langle Q^2 \rangle$  in a circular region  $A$  with radius  $R$  grows linearly with  $R$ , instead of with  $R^2$  as expected in the absence of screening; for long range screening  $\langle Q^2 \rangle$  grows faster than  $R$ : for a field whose autocorrelation function is the zero order Bessel function  $J_0$ ,  $\langle Q^2 \rangle \sim R \ln R$ . A  $J_0$  correlation function is not attainable in practice, but we show how to generate an optical field whose correlation function closely approximates this form; screening in such a field is well described by our theoretical results for  $J_0$ .  $\langle Q^2 \rangle$  can be measured by counting positive and negative singularities inside  $A$ , or more easily by counting signed zero crossings on the perimeter  $P$  of  $A$ . For the first method  $\langle Q^2 \rangle$  is calculated by integration over the charge correlation function  $C(r)$ , for the second by integration over the zero crossing correlation function  $\Gamma(r)$ . Using the explicit forms of  $C(r)$  and of  $\Gamma(r)$  we show that both methods of calculation yield the same result. We show that for short range screening the zero crossings can be counted along a straight line whose length equals  $P$ , but that for long range screening this simplification no longer holds. We also show that for realizable optical fields, for sufficiently small  $R$ ,  $\langle Q^2 \rangle \sim R^2$ , whereas for sufficiently large  $R$ ,  $\langle Q^2 \rangle \sim R$ . These universal laws are applicable to both short and pseudo-long range correlation functions.

## I. INTRODUCTION

Random (and other) paraxial optical fields generically contain numerous point topological singularities (defects) in a plane (the  $xy$ -plane) oriented perpendicular to the propagation direction ( $z$ -axis). These singularities, which include phase singularities (optical vortices) [1, 2] polarization singularities (C points) [3], gradient singularities (maxima, minima, and saddle points) [4], and curvature singularities (umbilic points) [5], are the defining features of the optical field, and are characterized by signed winding numbers (topological charges)  $q_{\pm}$ . Generically, vortices and gradient singularities have charge  $q_{\pm} = \pm 1$ , whereas the charge of C points and umbilic points is  $q_{\pm} = \pm 1/2$ .

Like electrostatic charges, topological charges screen one another: positive charges tend to be surrounded by a net excess of negative charge, and vice versa [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. A formal measure of screening is the charge correlation function  $C(\mathbf{r})$  [6], which by convention is constructed to measure the net excess of negative charge surrounding a positive charge at the origin. When

$$\iint_{-\infty}^{\infty} C(\mathbf{r}) d\mathbf{r} = -1 \quad \left( -\frac{1}{2} \text{ for C and umbilic points} \right) \quad (1)$$

screening is said to be complete, whereas for partial screening  $-1 < \iint_{-\infty}^{\infty} C(\mathbf{r}) d\mathbf{r} < 0$ . Here we concern ourselves with complete screening.

In generic optical fields the average net charge  $\langle Q \rangle = 0$ , but there are large fluctuations: these are characterized in lowest order by their variance  $\langle Q^2 \rangle$ . For a purely random collection of  $N$  singularities with say charges  $q_{\pm} = \pm 1$ ,  $\langle Q^2 \rangle \sim N$ . Screening significantly reduces these fluctuations, to an extent which depends on the rate of fall-off of  $C(r)$  for large  $r$ . In the generic case where  $C(r)$  decays sufficiently rapidly, canonically exponentially,  $\langle Q^2 \rangle \sim N^{1/2}$  [9]; this is the hallmark of short range screening. For a slowly decaying  $C(r)$  however, we shall see that the charge variance grows faster than  $N^{1/2}$ ; we call this regime long range screening.

$\langle Q^2 \rangle$  and  $C(\mathbf{r})$  are closely related, and given the latter the former can, in principle, be calculated. Here, we explore in detail the behavior of  $\langle Q^2 \rangle$  and its relationship to  $C(\mathbf{r})$  for long range screening. Specifically, we examine a particular example, first suggested in [10], and obtain analytical results for the behavior of  $\langle Q^2 \rangle$  in a large circular region of radius  $R$ , finding  $\langle Q^2 \rangle \sim R \ln R$ .

In elliptically polarized paraxial fields C points are vortices (zeros) of the right and left handed circularly polarized components of the field [3], so our results are applicable also to these singularities.  $C(\mathbf{r})$  is also known for the stationary and the umbilic points of random circular Gaussian fields, such as the real and imaginary parts of the optical field [15], but the correlation functions of these singularities differ from that of the vortices and our results are not necessarily applicable to these singularities.

Measurement of  $C(\mathbf{r})$  (either from experiment or computer simulation) requires locating and characterizing all vortices in each of a large number of independent realizations - this is a challenging task. On the other hand,  $Q$ , and therefore  $Q^2$ , can, using the index theorem [4], be obtained from measurements made only on the boundary of

the region of interest. As discussed in [9], a convenient way of accomplishing this is to count signed zero crossings of either the real or imaginary parts of the optical field. Correlations between these zero crossings are described by the zero crossing correlation function  $\Gamma(r)$ . In [9] it was argued that like  $C(\mathbf{r})$ , also  $\Gamma(r)$  can be used to calculate  $\langle Q^2 \rangle$ ; this argument was supported by numerical results and computer simulations, but was not demonstrated analytically.

The plan of this paper is as follows: In Section II we discuss the connection between  $\langle Q^2 \rangle$  and  $C(\mathbf{r})$  in a natural, i.e. sharply bounded, region of finite area - first reviewing known results for short range screening, and then presenting what are to our knowledge the first explicit results for such boundaries for long range screening<sup>1</sup>. These latter are the major contribution of this report. In Section III we discuss the connection between  $\langle Q^2 \rangle$  and the zero crossing correlation function  $\Gamma(r)$ , showing, analytically for the first time, the equivalence of the two seemingly different approaches to  $\langle Q^2 \rangle$ : one based on  $C(\mathbf{r})$ ; the other on  $\Gamma(r)$ . In Section IV we consider a practical realization of an optical field with long range correlations that could be used to compare experiments with the results presented here. In the discussion in Section V we consider asymptotic forms of  $\langle Q^2 \rangle$  for very small, and very large,  $R$ , for physically real optical fields. We summarize our main findings in the concluding Section VI.

## II. CHARGE VARIANCE IN A BOUNDED REGION AND THE CHARGE CORRELATION FUNCTION $C(r)$

In [9] the following quantitative relationship between the charge variance  $\langle Q^2 \rangle$  and the charge correlation function  $C(\mathbf{r})$  was derived for a bounded region of area  $A$  ([9], Eq. (37)) with  $\langle (\Delta N)^2 \rangle = \langle Q^2 \rangle$ :

$$\langle Q^2 \rangle = \langle N \rangle + \eta \iint_{-\infty}^{\infty} \mathcal{A}(\mathbf{r}) C(\mathbf{r}) d\mathbf{r}. \quad (2)$$

Here and throughout  $\langle \dots \rangle$  represents an ensemble average,  $N$  is the number of singularities with charges  $\pm 1$  contained in  $A$ ,  $\eta$  is the average number density of singularities, and  $\mathcal{A}(\mathbf{r})$  is the area of overlap between  $A$  and its replica displaced by  $\mathbf{r}$ .

In what follows we assume isotropy, and therefore take  $A$  to be a circular area with radius  $R$ . Elementary geometry yields

$$\mathcal{A}(\mathbf{r}) = \mathcal{A}(r) = \pi R^2 + B(r), \quad (3a)$$

$$\begin{aligned} B(r) &= -2R^2 \left( \arcsin\left(\frac{r}{2R}\right) + \frac{r}{2R} \sqrt{1 - \left(\frac{r}{2R}\right)^2} \right), \quad 0 \leq r \leq 2R, \\ &= 0, \quad r > 2R. \end{aligned} \quad (3b)$$

Noting that  $\langle N \rangle = \eta \pi R^2$ , Eq. (2) becomes

$$\langle Q^2 \rangle = \pi \eta R^2 \left[ 1 + 2\pi \int_0^{2R} r C(r) dr \right] + 2\pi \eta \int_0^{2R} r B(r) C(r) dr. \quad (4)$$

Eq. (4) is our basic starting point. We proceed to evaluate it for large  $R$ : first for a characteristic field with short range correlations; then for one whose correlations are long range.

### A. Charge correlation function

Halperin [6] first showed that assuming stationarity and circular Gaussian statistics, the charge correlation function for optical vortices can be written as

$$C(r) = - \frac{W'(r) [W(r)(W'(r))^2 + W''(r)(1 - W^2(r))]}{\pi W''(0)r(1 - W^2(r))^2}, \quad (5)$$

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<sup>1</sup> A long range screening result for a *Gauss-smoothed* boundary is given in [10].

where

$$W(r) = \langle E^*(0)E(r) \rangle / \langle |E(0)|^2 \rangle, \quad (6)$$

is the normalized autocorrelation function of the optical field  $E$ , and  $W'(r) = dW(r)/dr$ ,  $W''(r) = d^2W(r)/dr^2$ . Liu and Mazenko [7], and Berry and Dennis [10], subsequently noted that Eq. (5) can be cast into the convenient form

$$C(r) = \frac{-1}{2\pi r W''(0)} \frac{d\Omega^2(r)}{dr}, \quad (7)$$

where

$$\Omega(r) = \frac{W'(r)}{\sqrt{1 - W^2(r)}}. \quad (8)$$

The number density of vortices  $\eta$  also depends on  $W(r)$ , and is [2]

$$\eta = \frac{-W''(0)}{2\pi} \quad (9)$$

A useful measure of how rapidly screening sets in is

$$I_C(\rho) = 2\pi \int_0^\rho r C(r) dr \quad (10a)$$

$$= -1 - \frac{1}{W''(0)} \left[ \frac{(W'(\rho))^2}{1 - W^2(\rho)} \right]. \quad (10b)$$

For short range screening  $I \approx -1$  for  $\rho$  of order nearest neighbor separations, or greater; for long range screening  $I_C$  approaches  $-1$  only when  $\rho \rightarrow \infty$ .

## B. Short range screening

If  $C(r)$  decays sufficiently rapidly with  $r$ , the large  $r$  contribution in the first integral in Eq. (4) is negligible, and the upper limit can be extended to infinity. Using the screening relationship in Eq. (1) we then have

$$\langle Q^2 \rangle = 2\pi\eta \int_0^{2R} r B(r) C(r) dr. \quad (11)$$

In this same limit  $B(r)$  in Eq. (3b) can be replaced by its leading term in an expansion in powers of  $r/R$ ,  $B(r) \sim -2rR$ , and we have

$$\langle Q^2 \rangle = \frac{1}{4} \eta \Lambda_s P, \quad (12)$$

where  $P = 2\pi R$  is the length of the perimeter of  $A$ , and the screening length  $\Lambda_s$ , which is a measure of the width of  $I_C(\rho)$  in Eqs. (10), is [9]

$$\Lambda_s = -8 \int_0^\infty r^2 C(r) dr. \quad (13)$$

Thus, the hallmark of short range screening is that for  $R \gg \Lambda_s$ ,  $\langle Q^2 \rangle$  grows with the perimeter, rather than with the area, of  $A$ .

These results have a simple physical explanation [9]. Because charges deep inside  $A$  are perfectly screened they make no contribution to the net charge  $Q$ , and therefore no contribution to its variance  $\langle Q^2 \rangle$ . Charges located within a distance of order  $\Lambda_s$  from the boundary of  $A$ , however, are imperfectly screened. The number of these charges is  $n \approx \eta \Lambda_s P$ . For completely unscreened charges  $\langle Q^2 \rangle \approx n$ , whereas for partially screened charges we can expect  $\langle Q^2 \rangle \approx \alpha n$  [9], where from Eq. (12)  $\alpha = \frac{1}{4}$ .

Eqs. (12) and (13) also quantify the notion of short-range screening. The condition that  $\Lambda_s$  is finite is sufficient to insure that  $\langle Q^2 \rangle$  grows linearly with  $R$ . Thus, for short range screening  $C(r)$  (averaged over a period in the case that it oscillates) must fall faster than  $1/r^3$ .

### 1. Gaussian correlations

The canonical short range correlation function which gives rise to short range screening is the Gaussian, which we write as

$$W(v) = \exp(-\kappa^2 a^2 v^2), \quad (14a)$$

$$\kappa = 2\pi/(\lambda Z), \quad (14b)$$

where  $\lambda$  is the wavelength,  $Z$  is the (asymptotically large) distance between the random source and the screen on which the field  $E$  is measured, and  $v$  measures radial displacements on this screen.

$W(v)$  is the normalized autocorrelation function of the far field speckle pattern produced by a distribution of randomly phased sources with amplitudes

$$S(u) = \exp\left(-[u/(2a)]^2\right), \quad (15)$$

where  $u$  measures radial displacements in the source plane, and  $a$  is a measure of the width of the distribution<sup>2</sup>. The displacement  $\mathbf{r}$  and the displacement  $\mathbf{v}$  are related by the scale factor  $\kappa$

$$\mathbf{r} = \kappa \mathbf{v}, \quad (16)$$

giving for the Gaussian

$$W(r) = \exp(-a^2 r^2). \quad (17)$$

Then, using Eq. (13) we obtain

$$\Lambda_s = \zeta(3/2) / (\sqrt{2\pi}a) = 1.0422/a, \quad (18)$$

where  $\zeta$  is the Riemann zeta function.

The average spacing  $d$  between singularities is

$$d = \sqrt{1/\eta} = \sqrt{\pi}/a = 1.77/a, \quad (19)$$

which exceeds the screening length  $\Lambda_s$  by nearly a factor of two! This apparent paradox is resolved in Fig. 1, where it can be seen that the singularities tend to cluster with nearest neighbor spacings of order  $\Lambda_s$ .

A measure of how quickly screening sets in, thereby leading to local charge neutrality, is the rate of decay of  $I_C(\rho) + 1$ , Eqs. (10): for  $\rho = \Lambda_s$ ,  $I_C + 1 = 0.3$ , for  $\rho = 2\Lambda_s$ , it equals 0.003, and for  $\rho = 3\Lambda_s$ ,  $I_C + 1$  has dropped below  $3 \times 10^{-7}$ .

### 2. Smoothed boundaries

In ([10], Eq. (4.48) and (4.49)) a Gauss-smoothed boundary is assumed, and a *constant*, i.e.  $R$  independent, value for the charge variance, here  $\langle Q^2 \rangle_{const}$ , is obtained from

$$\langle Q^2 \rangle_{const} = \frac{N}{2} \left[ 1 + 2\pi\eta \int_0^\infty x \exp\left(-\frac{\pi x^2}{2A}\right) g_Q(x) dx \right], \quad (20a)$$

$$= \frac{1}{4} \int_0^\infty x \frac{W'(x)^2}{1 - W^2(x)} dx, \quad (20b)$$

where  $C(r)$  in [10] is our  $W(r)$ ,  $g_Q(x) = C(x)/\eta$  with  $C(x)$  given in Eq. (5),  $A$  is the area of the region of interest, here  $A = \pi R^2$ , and  $N = \eta A$  is the total number of charges inside this region. Eq. (20b) is obtained from Eq. (20a)

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<sup>2</sup> Such a source distribution can be obtained in practice by illuminating a random phase screen (typically a piece of finely ground glass) with a Gaussian laser beam.

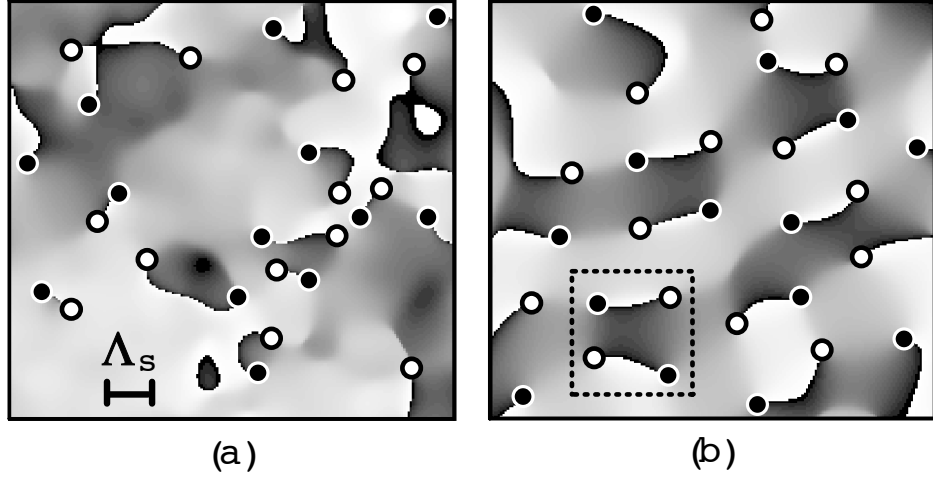


FIG. 1: Singularity structure. Shown is the phase of the optical field coded 0 to  $2\pi$  black to white for a field with (a) a Gaussian and (b) a  $J_0$  correlation function. Positive (negative) singularities are shown by filled white (black) circles. Both fields have the same average number density of singularities. (a) Gaussian correlation function. Here there are 13 positive and 13 negative singularities. The screening length  $\Lambda_s$  is shown by the bar. Note the clustering of singularities into small, charge-neutral groups with the spacing between nearest neighbor singularities being of order  $\Lambda_s$ , and with large empty regions between groups. It is this clustering that produces a screening length  $\sim 1/2$  the average spacing between singularities. (b)  $J_0$  correlation function. Here there are 14 positive and 14 negative singularities. As can be seen, there is a strong tendency for the singularities to be nearly equally spaced on a lattice-like structure in which each nearly square unit cell, shown by the dotted rectangle, contains two positive and two negative singularities. This structure exhibits substantial local charge neutralization, but because of defects in the lattice the overall screening is long range.

by using Eqs. (1), (7), (8), and (9), expanding  $\exp(-\pi x^2/(2A))$  keeping the leading term in  $x$ , and integrating by parts.

For the Gaussian correlation function in Eq. (17), Eq. (20b) can be evaluated analytically, and we obtain

$$\langle Q^2 \rangle_{const} = \frac{\pi^2}{48} = 0.2056... \quad (21)$$

This result shows that  $\langle Q^2 \rangle_{const}$  is not only independent of  $R$ , but also of  $a$ , and is therefore independent of the charge density  $\eta$  and all other system parameters! A question of considerable interest not discussed in [10] is whether or not this value for  $\langle Q^2 \rangle_{const}$  is an intrinsic property of the medium, or is it the result of a particular set of assumptions?

In [10] it is stated that Eq. (20a) is obtained from a calculation “with the boundary of  $A$  Gauss-smoothed to eliminate trivial edge effects”, but the details of the calculation are not given. In order to be able to more closely examine the physical content of Eq. (20a) we need a derivation of this equation. Here we briefly outline such a derivation, starting with the definition of a boundary that is “Gauss-smoothed”.

$\mathcal{A}(x)$  in Eq. (3) can be obtained from an area function  $\alpha(\mathbf{r})$  using

$$\mathcal{A}(\mathbf{x}) = 2 \int_0^\infty r dr \int_0^\pi d\theta [\alpha(\mathbf{r}) \alpha(\mathbf{r} + \mathbf{x})]. \quad (22)$$

Throughout this report we have used a disk shaped area described by the area function

$$\begin{aligned} \alpha(\mathbf{r}) &= \alpha(r) = 1, & 0 \leq r \leq R, \\ &= 0, & r > R. \end{aligned} \quad (23)$$

This function, shown in Fig. 2 as curve (a), inserted into Eq. (22) yields

$$\mathcal{A}(x) = 4 \int_{x/2}^R r dr \int_0^{\arccos(x/(2r))} d\theta, \quad (24)$$

which when evaluated yields Eq. (3).

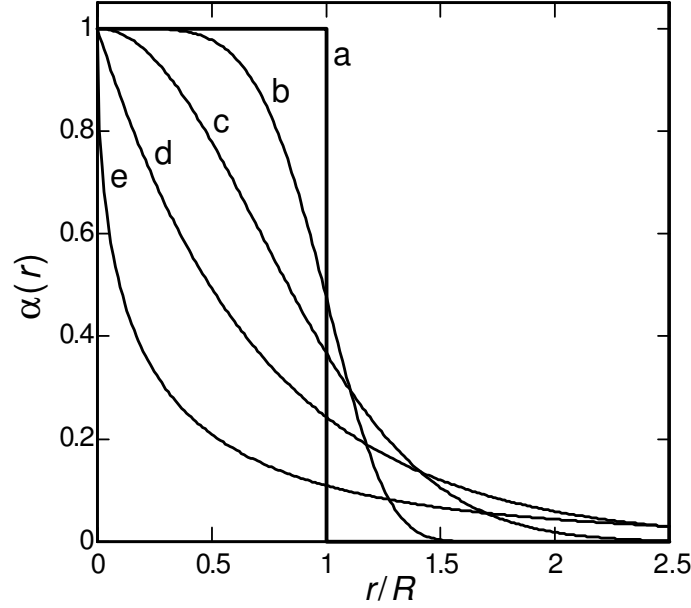


FIG. 2: Area functions. Shown is the normalized area function  $\alpha(r)$  for different values of  $p$  in Eq. (29). Curve a is the disk area function, Eq. (23), formally Eq. (29) with  $p \rightarrow \infty$ . Curves b - e are Eq. (29) with the following values for  $\int$ : curve b,  $p = 5$ , a so called super Gaussian; curve c,  $p = 2$ , a normal Gaussian; curve d,  $p = 1$ , a normal exponential; curve e,  $p = 1/2$ , a so called stretched exponential.

We note that in order to maintain the same number of charges  $N$  within  $A$  as there are for the disk, any other form for  $\alpha(r)$  must satisfy

$$2\pi \int_0^\infty r \alpha(r) dr = A = \pi R^2. \quad (25)$$

For a Gauss-smoothed boundary we have instead of Eq. (24),

$$\alpha(r) = \exp(-r^2/R^2) = \exp(-\pi r^2/A), \quad (26)$$

which is shown in Fig. 2 as curve (c). Inserting Eq. (26) into Eq. (22) yields

$$\mathcal{A}(x) = \frac{1}{2} A \exp\left(-\frac{\pi x^2}{2A}\right). \quad (27)$$

A peculiarity of Eq. (27) is that  $\mathcal{A}(0) = A/2$ , as opposed to  $\mathcal{A}(0) = A$ , the expected result, which is obtained from Eq. (3). As a result of this peculiarity, in the absence of screening the Gauss-smoothed calculation yields  $\langle Q^2 \rangle = N/2$ , instead of the expected random walk result  $\langle Q^2 \rangle = N$ .

Generalizing Eq. (4) to an arbitrary area function satisfying Eq. (25), we have

$$\langle Q^2 \rangle = \eta \mathcal{A}(0) + 2\pi\eta \int_0^\infty x \mathcal{A}(x) C(x) dx. \quad (28)$$

Inserting Eq. (27), and using  $C(x) = \eta g_Q(x)$  we recover Eq. (20a).

Now, if the physical content of  $\langle Q^2 \rangle_{const}$  were that it is an intrinsic property of the random medium, rather than simply the result of a calculation for a particular form for  $\alpha(r)$ , then other forms for  $\alpha(r)$  should yield the same result. In order to test this we redo the above calculation using a boundary smoothed by a general exponential function of the form,

$$\alpha(r) = \exp(-b_p (r/R)^p), \quad (29)$$

where  $0 < p < \infty$  is a real, not necessarily integer, number, and in order to satisfy Eq. (25),

$$b_p = \left[ \frac{2}{p} \Gamma\left(\frac{2}{p}\right) \right]^{(p/2)}. \quad (30)$$

Examples of  $\alpha(r)$  for different values of  $p$  are shown in Fig. 2 as curves (b), (d), and (e).

Inserting Eqs. (29) and (30) into (22), we find

$$\mathcal{A}(0) = 4^{-(1/p)} \pi R^2, \quad (31)$$

so that in the absence of screening  $\langle Q^2 \rangle = 4^{-(1/p)} N$ , which approaches  $N$  for large  $p$ , and approaches zero as  $p$  approaches zero.

For the charge variance itself we have

$$\langle Q^2 \rangle_{const} = p \left( \frac{\pi^2}{96} \right) \quad (32)$$

which reduces to the Gaussian result in Eq. (21) for  $p = 2$ , but differs from it for other values of  $p$ , demonstrating that  $\langle Q^2 \rangle_{const}$  is, in fact, not an intrinsic property of the medium, but is dependent on the particular choice of  $\alpha(r)$ .

In [9] a short range autocorrelation function of the form

$$W(r) = 2J_1(ar)/(ar) \quad (33)$$

was used. For this form for arbitrary  $p$  we obtain by numerical integration

$$\langle Q^2 \rangle_{const} = p(0.168077...), \quad (34)$$

which is again independent of  $a$ .

Because both Eqs. (32) and (34) are continuous functions of  $p$ , a particular value for  $\langle Q^2 \rangle_{const}$  does not uniquely specify neither the field autocorrelation function nor the boundary smoothing function. For example,  $\langle Q^2 \rangle_{const} = 0.2056...$  for a Gaussian autocorrelation function and a Gaussian boundary smoothing function for which  $p = 2$ . But the exact same value for  $\langle Q^2 \rangle_{const}$  is obtained using the  $J_1$  autocorrelation function in Eqs. (33) and a boundary smoothing function with  $p = 1.1817...$

### C. Long Range Screening

Berry and Dennis [10] have introduced the long range correlation function

$$W(r) = J_0(ar), \quad (35)$$

where  $J_n$  is a Bessel function of integer order  $n$ . This form for  $W(r)$  arises from a random source distribution  $S(u)$  that takes the form of a very thin ring,

$$S(u) = \delta(u - a). \quad (36)$$

Practical realization of a  $J_0$  correlation function is discussed in Section IV.

Using Eqs. (10) with  $\rho \rightarrow \infty$  the screening relationship in Eq. (1) is easily verified, whereas upon inserting Eqs. (7) and (8) into Eq. (13) it is immediately seen that  $\Lambda_s$  does not converge. These are the conditions for long range screening.

Inserting Eqs. (3b), (7), and (8) into Eq. (4), and integrating by parts, we obtain

$$\langle Q^2 \rangle = \frac{1}{2\pi} \int_0^{2R} \sqrt{4R^2 - r^2} \frac{(W'(r))^2}{1 - W^2(r)} dr. \quad (37)$$

In the remainder of this report we use this highly convenient form for  $\langle Q^2 \rangle$  exclusively.

For the case of  $J_0$ , Eq. (35), we break the integral on the R.H.S. of Eq. (37) into two parts:

$$\langle Q^2 \rangle = \mathfrak{I}_1 + \mathfrak{I}_2, \quad (38)$$

where

$$\mathfrak{I}_1 = \frac{a^2}{2\pi} \int_0^\Delta dr \sqrt{4R^2 - r^2} \frac{J_1^2(ar)}{1 - J_0^2(ar)}, \quad (39a)$$

$$\mathfrak{I}_2 = \frac{a^2}{2\pi} \int_\Delta^{2R} dr \sqrt{4R^2 - r^2} \frac{J_1^2(ar)}{1 - J_0^2(ar)}, \quad (39b)$$

with  $1/a \ll \Delta \ll R$ .

In evaluating  $\mathfrak{J}_1$  and  $\mathfrak{J}_2$  we will need the asymptotic forms

$$\frac{J_1^2(ar)}{1 - J_0^2(ar)} \approx J_1^2(ar) \approx \frac{2 \cos^2(ar + \pi/4)}{\pi ar}; \quad r \gg 1/a. \quad (40)$$

Expanding  $\sqrt{4R^2 - r^2}$  and keeping only the leading  $R$  term we have for  $\mathfrak{J}_1$ ,

$$\mathfrak{J}_1 \approx \frac{aR}{\pi} \int_0^{a\Delta} dx \frac{J_1^2(x)}{1 - J_0^2(x)}. \quad (41)$$

This form diverges as  $a\Delta \rightarrow \infty$ , and so we regularize it by writing

$$\mathfrak{J}_1 \approx \frac{aR}{\pi} \int_0^{a\Delta} dx \left[ \frac{J_1^2(x)}{1 - J_0^2(x)} - J_1^2(x) + J_1^2(x) \right], \quad (42a)$$

$$\approx \frac{aR}{\pi} \left[ \mathcal{D} + \int_0^{a\Delta} dx J_1^2(x) \right], \quad (42b)$$

where

$$\mathcal{D} = \int_0^\infty dx \frac{J_0^2(x) J_1^2(x)}{1 - J_0^2(x)} = 0.5630468586... \quad (43)$$

is evaluated numerically. Using Eq. (40) the remaining integral in Eq. (42b) can be evaluated analytically in the limit of large  $a\Delta$  as follows:

$$\int_0^{a\Delta} J_1^2(x) dx = \lim_{\epsilon \rightarrow 0^+} \int_0^{a\Delta} dx x^{-\epsilon} J_1^2(x), \quad (44a)$$

$$= \lim_{\epsilon \rightarrow 0^+} \left[ \int_0^\infty dx x^{-\epsilon} J_1^2(x) - \int_{a\Delta}^\infty dx x^{-\epsilon} J_1^2(x) \right], \quad (44b)$$

$$\approx \lim_{\epsilon \rightarrow 0^+} \left[ \frac{\Gamma(\frac{3+\epsilon}{2})}{2^\epsilon \Gamma(2) \Gamma(\frac{1+\epsilon}{2})} {}_2F_1\left(\frac{3-\epsilon}{2}, \frac{1-\epsilon}{2}; 2; 1\right) - \int_{a\Delta}^\infty x^{-\epsilon} \frac{1}{\pi x} \right], \quad (44c)$$

$$= \lim_{\epsilon \rightarrow 0^+} \left[ \frac{\Gamma(\frac{3-\epsilon}{2})}{2^\epsilon \Gamma(2) \Gamma(\frac{1+\epsilon}{2})} \frac{\Gamma(2) \Gamma(\epsilon)}{\Gamma(\frac{1+\epsilon}{2}) \Gamma(\frac{3+\epsilon}{2})} - \frac{1}{\epsilon \pi} (a\Delta)^{-\epsilon} \right], \quad (44d)$$

$$= \frac{\gamma + 3 \ln 2 - 2}{\pi} + \frac{1}{\pi} \ln(a\Delta), \quad (44e)$$

where  $\gamma = 0.5772...$  is Euler's constant.

Similarly, in calculating  $\mathfrak{J}_2$ , Eq. (39b), we use the asymptotic form in Eq. (40) to obtain

$$\mathfrak{J}_2 \approx \frac{a}{2\pi^2} \int_\Delta^{2R} dr \frac{\sqrt{4R^2 - r^2}}{r}, \quad (45a)$$

$$\approx \frac{a}{2\pi^2} \left[ \sqrt{4R^2 - x^2} - 2R \ln \frac{4R^2 + 2R\sqrt{4R^2 - x^2}}{x} \right]_\Delta^{2R}, \quad (45b)$$

$$\approx \frac{a}{2\pi^2} \left[ -2R \ln \frac{4R^2}{2R} - 2R + 2R \ln \frac{8R^2}{\Delta} \right], \quad (45c)$$

$$= \frac{aR}{\pi^2} \left[ \ln \frac{4R}{\Delta} - 1 \right]. \quad (45d)$$

Combining  $\mathfrak{J}_1$  and  $\mathfrak{J}_2$ , we obtain the central result of this report,

$$\boxed{\langle Q^2 \rangle = \frac{aR}{\pi^2} [K + \ln(aR)]}, \quad (46a)$$



where

$$K = \pi\mathcal{D} + \gamma + 5 \ln 2 - 3 = 2.81181544... \quad (46b)$$

We note that for rather modest values  $aR > 17$  the  $\ln(aR)$  term dominates, so that this term should be easily accessible to experiment. Direct numerical integration of Eq. (2) reveals that Eq. (46a) is good to better than 3% for  $aR = 2$ , to better than 0.3% for  $aR = 10$ , and that thereafter the percentage error decreases as  $\sim 3/(aR)$ .

As for the Gaussian, a measure of how quickly (or slowly) screening sets in is the average rate of decay of  $I_C(\rho) + 1$ . For large  $\rho$  this average rate is  $\sim 2/(\pi a \rho)$ , showing that although screening is ultimately complete, it approaches completeness very slowly.

### 1. Smoothed boundaries

What does boundary smoothing do to  $\langle Q^2 \rangle$  when the correlations are long ranged?

For the long range correlation function  $J_0(ar)$ , [10] states “we can show from (4.48) [here Eq. (20a)] that ...  $\langle Q^2 \rangle \sim \sqrt{N}$ ”, i.e. for a circular region of radius  $R$ ,  $\langle Q^2 \rangle \sim R$ .

For large  $R$ , the limit also used in [10], we find here for all generalized exponential smoothing functions in Eq. (29).

$$\langle Q^2 \rangle = \Xi(p) aR. \quad (47)$$

For the Gaussian smoothing function used in [10] we obtain analytically

$$\Xi(2) = \frac{1}{4\sqrt{2\pi}} = 0.09974. \quad (48)$$

In general, however,  $\Xi(p)$  must be evaluated numerically. Introducing scaled variables  $\tilde{x} = b_p^{1/p} x/R$ ,  $\tilde{r} = b_p^{1/p} r/R$ , and using the asymptotic form  $\Omega^2(x) \approx 1/(\pi x)$ , Eq. (40), we have

$$\Xi(p) = \frac{-1}{2\pi^2 b_p^{1/p}} \int_0^\infty d\tilde{x} \left( \frac{d\mathcal{A}(\tilde{x})}{d\tilde{x}} \right), \quad (49a)$$

$$\mathcal{A}(\tilde{x}) = 2 \int_0^\infty \tilde{r} d\tilde{r} \int_0^\pi d\theta \exp(-\tilde{r}_+^p - \tilde{r}_-^p), \quad (49b)$$

$$\tilde{r}_\pm = \sqrt{\tilde{r}^2 + \tilde{x}^2/4 \pm \tilde{r}\tilde{x} \cos \theta}. \quad (49c)$$

Carrying out the required numerical integrations we obtain:  $\Xi(4) = 0.1615$ ;  $\Xi(6) = 0.2052$ ;  $\Xi(8) = 0.2338$ ; and  $\Xi(10) = 0.2561$ .

Thus, also for long range correlations the results of boundary smoothing change when the arbitrary smoothing function is changed.

## III. CHARGE VARIANCE IN A BOUNDED REGION AND THE ZERO CROSSING CORRELATION FUNCTION $\Gamma$

As indicated in the Introduction,  $Q$ , and therefore  $\langle Q^2 \rangle$ , can be obtained from measurements made only on the boundary of the region of interest  $A$ , here a circle of radius  $R$ . The method suggested in [9], which is amenable to calculation, is the following: Signed zero crossings (ZCs) of either the real  $\mathcal{R}$ , or of the imaginary  $\mathcal{I}$ , parts of the wave functions that cross the boundary are counted, with each positive (negative) ZC contributing  $+\frac{1}{2}$  ( $-\frac{1}{2}$ ) to  $Q$ ; the sign  $\sigma_{\mathcal{R}}$  of a zero crossing of  $\mathcal{R}$  is  $\sigma_{\mathcal{R}} = \text{sign}(-\mathcal{I}\mathcal{R}_s)$ , that of a ZC of  $\mathcal{I}$  is  $\sigma_{\mathcal{I}} = \text{sign}(\mathcal{R}\mathcal{I}_s)$ , where  $\mathcal{R}_s = \partial\mathcal{R}/\partial s$ ,  $\mathcal{I}_s = \partial\mathcal{I}/\partial s$  [9].

Writing the lineal number density of positive (negative) zero crossings of  $\mathcal{R}$  or of  $\mathcal{I}$  as  $n_+, (n_-)$ , their sum and difference are

$$n_0 = n_+ + n_-, \quad (50a)$$

$$\Delta N = n_+ - n_-. \quad (50b)$$

where, for circular Gaussian statistics [19]

$$n_0 = \frac{1}{\pi} \sqrt{-W''(0)}. \quad (51)$$

For a straight line of length  $L$  oriented along say the  $x$ -axis [9],

$$\langle (\Delta N)^2 \rangle_L = \frac{1}{4} n_0 L \left[ 1 + 2 \int_0^L \Gamma(\Delta x) d(\Delta x) - \frac{2}{L} \int_0^L |\Delta x| \Gamma(\Delta x) d(\Delta x) \right]. \quad (52)$$

The zero crossing correlation function  $\Gamma$  is

$$\Gamma(\Delta x) = -\frac{1}{\pi^2 n_0} \frac{(1 - W^2(\Delta x) W''(\Delta x) + W(\Delta x) (W'(\Delta x))^2) \sin^{-1} W(\Delta x)}{(1 - W^2(\Delta x))^{3/2}} \quad (53a)$$

$$= \frac{-1}{\pi^2 n_0} \arcsin(W(\Delta x)) \frac{d\Omega(\Delta x)}{d(\Delta x)}, \quad (53b)$$

where  $\Omega$  is obtained from Eq. (8) with  $r$  replaced by  $\Delta x$ .  $\Gamma(\Delta x)$  and  $C(r)$  are compared in Fig. 3.

For a circle of radius  $R$ ,

$$\begin{aligned} \langle (\Delta N)^2 \rangle_C &= \langle Q^2 \rangle, \\ &= \frac{1}{4} n_0 (2\pi R) \left[ 1 + 2 \int_0^\pi \Gamma(\Delta\theta) d(\Delta\theta) \right], \end{aligned} \quad (54)$$

where  $\Delta\theta$  measures the angular separation of two points on the rim of the circle. This form for  $\langle (\Delta N)^2 \rangle$  follows from Section F of [9] with  $x$  replaced by  $\theta$ . The corresponding form for  $\Gamma(\Delta\theta)$  is

$$\Gamma(\Delta\theta) = \frac{-1}{\pi^2 n_0^{(\theta)}} \arcsin(W) \frac{d\Omega(\Delta\theta)}{d(\Delta\theta)}, \quad (55)$$

where  $n_0^{(\theta)}$  is the number of zero crossings per unit of arc that cross the rim of the circle, and  $\Omega(\Delta\theta)$  is obtained from Eq. (53b) with  $\Delta x$  replaced by  $\Delta\theta$ . This form for  $\Gamma(\Delta\theta)$  follows from Section G of [9] with  $x$  replaced by  $\theta$ .

Using the fact that in the limit  $\Delta\theta$  goes to zero the difference between the rim (arc) of the circle  $R\Delta\theta$  and the corresponding chord, here  $r$ , vanishes, together with  $n_0^{(\theta)} |d\theta| = n_0 |dr|$ , we have

$$n_0^{(\theta)} = R n_0, \quad (56)$$

where  $n_0$ , given in Eq. (50a), is the lineal number density of zero crossings.

In what follows we refer to Eq. (52) as the *linear*  $\Gamma$  formula, and to Eq. (54) as the *circular*  $\Gamma$  formula.

The field autocorrelation function  $W(\Delta\theta)$  appearing in  $\Omega(\Delta\theta)$  depends only on the straight line separation, i.e. chord length  $r$ , between two points on the circle rim separated by  $\Delta\theta$ ,

$$W(\Delta\theta) = W(r), \quad (57a)$$

$$r = 2R \sin(\Delta\theta/2). \quad (57b)$$

### A. Exact Equivalence

We now proceed to show explicitly that  $\langle Q^2 \rangle$  in Eq. 2, which is written in terms of the charge correlation function  $C$  and the area of  $A$ , and  $\langle Q^2 \rangle$  in the circular  $\Gamma$  formula, Eq. (54), which is written in terms of the zero crossing correlation function  $\Gamma$  and the perimeter of  $A$ , are identical - as they must be.

Returning to Eq. (37) we write

$$\langle Q^2 \rangle = \frac{1}{2\pi} \int_0^{2R} dr \sqrt{4R^2 - r^2} \frac{W'(r)}{\sqrt{1 - W^2(r)}} \frac{d}{dr} \sin^{-1} W(r), \quad (58a)$$

$$= \frac{1}{2\pi} \left\{ \pi^2 R n_0 - \int_0^{2R} dr \sin^{-1} W(r) \frac{d}{dr} \left[ \sqrt{4R^2 - r^2} \frac{W'(r)}{\sqrt{1 - W^2(r)}} \right] \right\}. \quad (58b)$$

We now compare this with the circular  $\Gamma$  formula, Eq. (54). Writing

$$\Gamma(\Delta\theta) = -\frac{1}{\pi^2 n_0^{(\theta)}} \sin^{-1} W(r) R \cos(\Delta\theta/2) \frac{d}{dr} \left[ (1 - W^2(r))^{-1/2} R \cos(\Delta\theta/2) \frac{d}{dr} W(r) \right], \quad (59)$$

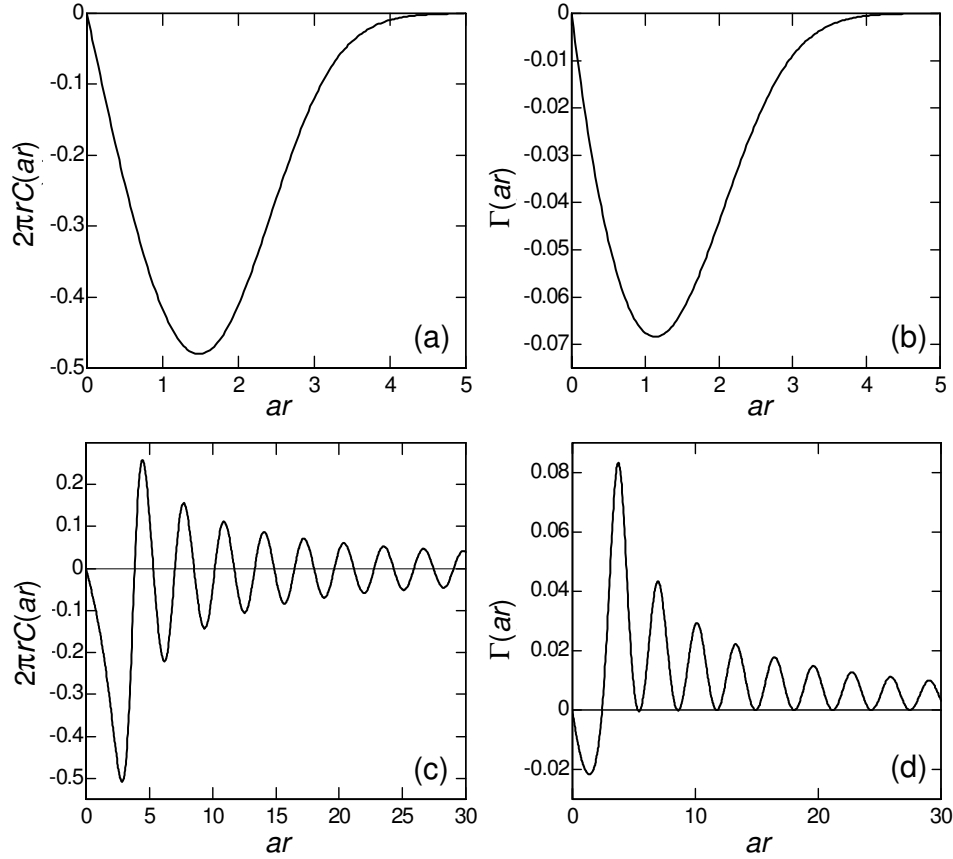


FIG. 3: Correlation Functions. (a,b) Short range correlations. (a) Charge correlation function  $C(ar)$  and (b) Zero crossing correlation function  $\Gamma(ar)$  for short range Gaussian field correlator  $W(ar) = \exp(-a^2 r^2)$ . (c,d) Long range correlations. (c) Charge correlation function  $C(ar)$  and (d) Zero crossing correlation function  $\Gamma(ar)$  for long range field correlator  $W(ar) = J_0(ar)$ . Short range correlations:  $C(ar)$  in (a) is everywhere negative, indicating that on average positive (negative) charges are surrounded by a tight cloud of excess negative (positive) charge, leading to local charge neutralization - this is the characteristic field structure of short range screening, see Fig. 1(a). Similarly,  $\Gamma(ar)$  in (b) is also everywhere negative, indicating that a positive (negative) zero crossing on the boundary is likely to be followed by an excess of negative (positive) zero crossings that quickly cancel its contribution to the net charge. This structure leads to a linear increase of the charge variance with boundary length. Long range correlations: Although for, say, a central positive charge the first negative dip in  $C(ar)$  in (c) integrates to an excess negative charge of  $-0.76$ , indicating substantial, albeit incomplete, local charge neutralization, much of this is offset by the following positive peak. As can be seen, the net average field structure consists of rings whose weak excess charge alternates positive/negative, with each successive negative ring having marginally greater area, and therefore marginally greater negative charge, than the positive ring that follows it. These oscillating cancellations give rise to a slow  $1/r$  approach to complete screening that is the hallmark of long range screening. This weak structure is too diffuse to be seen in Fig. 1(b). For the zero crossing correlation function  $\Gamma(ar)$  in (d), a positive (negative) zero crossing on the boundary is likely to be followed by a long string of zero crossings with a weak excess of positive (negative) signs that reflect the bulk ring structure (the boundary runs through one of these rings). This structure leads to an  $R \ln R$  growth of the charge variance.

and using the fact that  $0 \leq \Delta\theta \leq \pi$ , so that  $\cos(\Delta\theta/2) \geq 0 = \sqrt{4R^2 - r^2}/(2R)$ , we have

$$\begin{aligned} \langle (\Delta N)^2 \rangle_C &= \frac{n_o \pi R}{2} - \frac{1}{\pi} \int_0^{2R} \frac{dr}{R \cos(\Delta\theta/2)} R \cos(\Delta\theta/2) \sin^{-1} W(r) \\ &\quad \times \frac{d}{dr} \left[ (1 - W^2(r))^{-1/2} R \cos(\Delta\theta/2) \frac{dW(r)}{dr} \right], \end{aligned} \quad (60a)$$

$$= \frac{n_o \pi R}{2} - \frac{1}{2\pi} \int_0^{2R} dr \sin^{-1} W(r) \frac{d}{dr} \left[ \sqrt{4R^2 - r^2} \frac{W'(r)}{\sqrt{1 - W^2(r)}} \right], \quad (60b)$$

*Q.E.D.*

## B. Approximate Equivalences

It was suggested in [9] that the zero crossings of  $\mathcal{R}$  or of  $\mathcal{I}$  need not necessarily be counted over the actual perimeter  $P$  of  $\mathcal{A}$ , but rather that in an isotropic system one could perform the count over any straight line with the same length as  $P$ . This suggestion, supported by numerical analysis and computer simulations, was based on the notion that for short range correlations the geometry of the line - rectangular, circular, straight, etc. - was unimportant, as long as over most of its length its radius of curvature was large compared to the screening length  $\Lambda_s$ . Here we demonstrate the validity of this suggestion analytically for short range correlations, and examine its validity for long range correlations.

We define an error parameter  $\mathcal{E}$  by

$$\langle(\Delta N)^2\rangle_L = \langle Q^2\rangle + \mathcal{E}, \quad (61a)$$

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3, \quad (61b)$$

where

$$\mathcal{E}_1 = -\frac{n_o}{2} \int_0^{2\pi R} dr r \Gamma(r), \quad (62)$$

$$\mathcal{E}_2 = n_o \pi R \int_{2R}^{2\pi R} dr \Gamma(r), \quad (63)$$

$$\mathcal{E}_3 = \frac{1}{2\pi} \int_0^{2R} dr \sin^{-1} W(r) \frac{d}{dr} \left[ \left( \sqrt{4R^2 - r^2} - 2R \right) \frac{W'(r)}{\sqrt{1 - W^2(r)}} \right]. \quad (64)$$

### 1. Short range correlations

For large  $R$ , when  $W$  is short ranged  $\Gamma(r)$  decays to zero well before the upper limit of the integral on the R.H.S. of Eq. (62) is reached. Extending this limit to  $\infty$ , it is easily seen that  $\mathcal{E}_1$  is a constant that does not grow with  $R$ , and can therefore be neglected; for the same reasons,  $\mathcal{E}_2$  in Eq. (63) can similarly be neglected. Expanding the square root in the R.H.S. of Eq. (64), it is easily seen that also  $\mathcal{E}_3$  does not grow with  $R$ , so it too can be neglected. Thus for short range correlations the linear  $\Gamma$  formula and the exact result for  $\langle Q^2 \rangle$  are equivalent in the limit of large  $R$ .

This equivalence can be demonstrated more directly. Returning to Eq. (37), we have to leading order in  $r/R$

$$\langle Q^2 \rangle \approx \frac{R}{\pi} \int_0^\infty dr \frac{(W'(r))^2}{1 - W^2(r)}, \quad (65a)$$

$$= \frac{R}{\pi} \int_0^\infty dr \frac{W'(r)}{\sqrt{1 - W^2(r)}} \frac{d}{dr} \arcsin(W(r)), \quad (65b)$$

$$= \frac{n_o \pi R}{2} \left[ 1 + 2 \int_0^\infty dr \Gamma(r) \right], \quad (65c)$$

which is also the leading, i.e.  $R$  dependent, term in the linear  $\Gamma$  formula, Eq. (52).

We therefore conclude, in full accord with [9], that for short range screening the linear  $\Gamma$  formula can, indeed, be used to measure  $\langle Q^2 \rangle$ .

### 2. Long range correlations

For long range correlations we need the asymptotic form of  $\Gamma(r)$ , which for  $J_0$  is

$$\Gamma(r) \approx \frac{a}{\pi^3 n_0} \frac{1 + \sin(2ar)}{r}. \quad (66)$$

To leading order the oscillating  $\sin(2ar)/r$  term makes no contribution, and we have:

for  $\mathcal{E}_1$ ,

$$\mathcal{E}_1 \approx - \int_0^{2\pi R} dr r \frac{a}{2\pi^3 r}, \quad (67a)$$

$$= \frac{aR}{\pi^2}, \quad (67b)$$

for  $\mathcal{E}_2$ ,

$$\mathcal{E}_2 \approx R \int_{2R}^{2\pi R} dr \frac{a}{\pi^2 r}, \quad (68a)$$

$$= \frac{aR}{\pi^2} \ln \pi, \quad (68b)$$

and for  $\mathcal{E}_3$ ,

$$\mathcal{E}_3 \approx -\frac{1}{2\pi} \int_0^{2R} dr W(r) W''(r) \left( \sqrt{4R^2 - r^2} - 2R \right), \quad (69a)$$

$$\approx -\frac{1}{2\pi} \int_0^{2R} dr \left( -\frac{a}{\pi r} \right) \left( \sqrt{4R^2 - r^2} - 2R \right), \quad (69b)$$

$$= -\frac{aR}{\pi^2} (\ln 2 - 1). \quad (69c)$$

where in evaluating Eq. (69a) we use the asymptotic form

$$W(r) W''(r) \approx -\frac{2a}{\pi r} \sin^2(ar). \quad (70)$$

Assembling the pieces yields

$$\mathcal{E} = \frac{aR}{\pi^2} \ln \left( \frac{2}{\pi} \right) = -.0458aR. \quad (71)$$

Thus, unlike the case of short range screening, for long range screening the linear  $\Gamma$  formula is always larger than the true result; the percentage error, however, is not severe, being 8.8% for  $aR = 10$ , 6.3% for  $aR = 100$ , and 4.9% for  $aR = 1000$ .

$\langle (\Delta N)^2 \rangle_L$  exceeds  $\langle Q^2 \rangle$  because setting  $L = 2\pi R$  in the integrals over  $\Gamma(\Delta x)$  in Eq. (52) implies that the correlations between zero crossings act along the arc of the circle. But, as already noted in connection with the circular  $\Gamma$  formula, these correlations act only along the chords of the circle. However, except for the diameter, all chords are shorter than their arcs, so that  $L = 2\pi R$  necessarily overestimates the correlation effects.

We now inquire as to what value of  $L$  in the upper limits of the correlation integrals appearing on the R.H.S. of Eq. (52) establishes equality between  $\langle (\Delta N)^2 \rangle_L$  and  $\langle Q^2 \rangle$ . Writing  $L = 2pR$ , we have

$$\langle (\Delta N)^2 \rangle_L = \langle Q^2 \rangle + faR, \quad (72a)$$

$$f = \frac{\ln(p/2) + 1 - p/\pi}{\pi^2}. \quad (72b)$$

Setting  $f = 0$  and solving for  $p$  we obtain

$$p = -\pi \text{LW} \left( -\frac{2}{\pi e} \right) = 1.01701687... \quad (73)$$

where LW is the LambertW function [20], and  $e = 2.718...$  is the base of the natural logarithm. Eq. (73) suggests that as a physically attractive approximation  $L$  should be set equal to the longest chord,  $2R$ , rather than to the longest arc,  $2\pi R$ . Indeed, inserting  $p = 1$  into  $f$  yields the excellent approximation  $\mathcal{E} = -0.00116aR$  - a forty-fold improvement over Eq. (71).

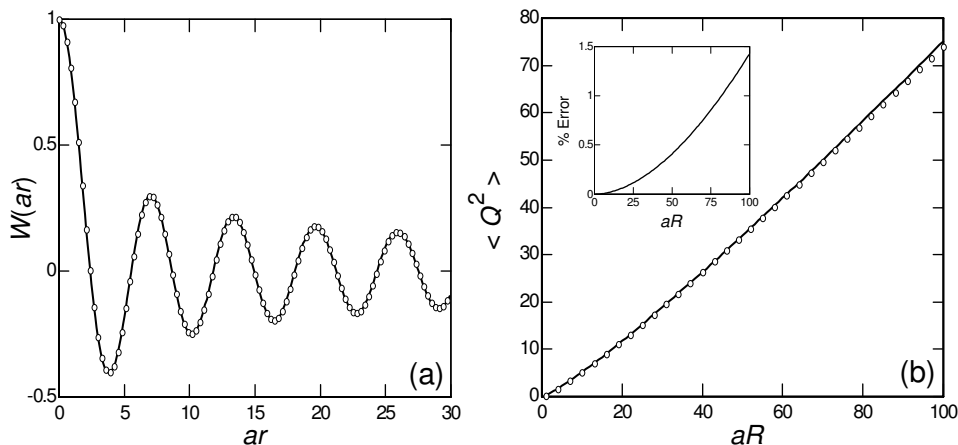


FIG. 4: Practical realization of  $J_0$ . (a) Correlation functions. Solid curve  $W(ar) = J_0(ar)$ , small circles  $W(ar)$  equal to the practical realization of  $J_0$ , Eq. (74a); in both cases  $a = 1$  and  $b = 0.01$ . The agreement between the two forms is excellent out to  $ar = 100$ . (b)  $\langle Q^2 \rangle$  vs.  $aR$ . Solid curve the exact result, Eq. (37) with  $J_0$  correlation function, small circles Eq. (37) with practical realization of  $J_0$ . The inset shows the percentage error obtained using the practical realization. This error is less than 0.5% for  $aR = 50$ , and less than 1.5% for  $aR = 100$ .

#### IV. PRACTICAL REALIZATION OF $J_0$

As discussed in Section IIc, the long range correlation function  $W_{J_0}(r) = J_0(ar)$  arises from a source  $S(u)$  that takes the form of a ring of zero width, Eq. (36). Here we consider a ring of mean radius  $a$  and finite width  $b$  as a practical realization,  $W_{pJ_0}(r)$ , of  $W_{J_0}(r)$ .

We obtain  $W_{pJ_0}(r)$  from the Van Cittert-Zernike theorem [21] as

$$W_{pJ_0}(r) = \frac{1}{abr} [(a + b/2) J_1(ar + br/2) - (a - b/2) J_1(ar - br/2)], \quad (74a)$$

$$\approx J_0(ar) \left[ 1 - \frac{1}{6} \left( \frac{b}{2a} \right)^2 (ar)^2 \right] - \frac{1}{6} \left( \frac{b}{2a} \right)^2 (ar) J_1(ar) + \mathcal{O} \left( \left( \frac{b}{2a} \right)^4 \right). \quad (74b)$$

$W_{pJ_0}(r)$  closely matches  $W_{J_0}(r)$  over the region  $0 \leq ar \leq a/b$ , see Fig. 4(a). We can therefore expect for  $0 \leq ar \leq \frac{1}{2}a/b$ , and in fact do obtain, see Figs. 4(b,c), a close match between the result obtained using  $W_{pJ_0}$  in Eqs. (7) and (8) and integrating Eq. (37) numerically, and that obtained using  $W_{J_0}$ .

A ratio  $b/a = 100$  appears to be experimentally practical, so that the long range screening results presented here should be amenable to experimentation.

#### V. DISCUSSION

##### A. Smoothed boundaries

What does smoothing the boundary of the wavefield mean both physically and mathematically? Close examination of the derivation of Eq. (20a) reveals that *all* boundary smoothing functions act as filters that smoothly reduce to zero *not* the field amplitude as one recedes from the central point, but rather the *number density of singularities*.

How can such a fall-off be achieved?

A physically meaningful possibility is to devise a source function  $S$  that leads to a wavefield with an intrinsic singularity density that falls smoothly to zero from some central point. However, not only is there no suitable source function known, even in principle, but also such a wavefield, if it could be created, would violate a fundamental assumption in the derivation of the charge correlation function  $C(r)$  and in the calculation of  $\langle Q^2 \rangle$  that is used in [10] as well as here, namely that the system is stationary, i.e. that on average it is the same everywhere, that there is no central point. Thus, for such a system the results given in [10], and our extension of them, need not necessarily apply.

So, if an intrinsic average number density of singularities that is independent of position is needed, then how can the fall-off required by the smooth boundary in [10] be obtained? The only possibility that comes to mind is the

following: Some arbitrary point, the central point, is surrounded by a small circle and a series of narrow concentric rings. Suitable fractions of the singularities in these rings are then discarded to create a singularity density profile that approximates, say, the Gaussian in [10], or any other desired smoothing function.

Of course, an experimentalist would object that not only is such a procedure entirely arbitrary, but also that no good could ever come from throwing away data. And, as shown above, she would be right, because the cost of boundary smoothing for short range correlations is a loss of all wavefield information. Similarly, for long range correlations the magnitude of the calculated fluctuations change with changes in smoothing function, showing that the coefficient  $\Xi$  in Eq. (47) cannot be interpreted as a screening length, as is done in Eq. (12).

## B. Deviations From the Linear Law

### 1. Small $R$ for any correlation function

Surprisingly, perhaps, for sufficiently small  $R$ ,  $\langle Q^2 \rangle \sim R^2$  for all field correlation functions for which  $W'(0) = 0$ ; this includes not only the Gaussian in Eq. (17),  $J_0$  in Eq. (35), the practical approximation to  $J_0$  in Eq. (74a), but also every other physically realizable short and long range correlation function.  $W'(0) = 0$  follows from expansion of the Hankel transform in the Van Cittert-Zernike theorem [21], and the fact that all physically realizable  $S$  are bounded. The proof of this universal  $R^2$  law for  $R < 1/a$  is as follows: Returning to Eq. (37) we have for  $ar \ll 1$ ,

$$\langle Q^2 \rangle = \eta \int_0^{2R} dr \sqrt{4R^2 - r^2}, \quad (75a)$$

$$= \eta \pi R^2, \quad (75b)$$

$$= N. \quad (75c)$$

The physical meaning of this result is that for sufficiently small  $A = \pi R^2$  there can be no screening because the probability of finding the required negative charge within  $A$  is vanishingly small.

### 2. Large $R$ for long range correlation functions that decay slower than $J_0$

(i) If one had a form for  $W(r)$  for which  $W'(r)$  decays asymptotically as  $r^{-\beta}$  with  $\beta < \frac{1}{2}$ , then

$$\langle Q^2 \rangle \approx \frac{\Gamma(\frac{1}{2} - \beta)}{2^{2\beta+1} \sqrt{\pi} \Gamma(2 - \beta)} R^{2-2\beta}. \quad (76)$$

(ii) For  $\beta = \frac{1}{2}$ ,  $\langle Q^2 \rangle \approx R \ln R$ , Eq. (46a).

(iii) For  $\beta > \frac{1}{2}$ ,  $\langle Q^2 \rangle \approx R$ , Eq. (12).

The proofs of assertions (i) and (iii) are as follows:

For  $\beta > \frac{1}{2}$ , Eq. (13) converges, and therefore Eq. (12) holds.

For  $\beta < \frac{1}{2}$  we rewrite Eq. (37) as

$$\langle Q^2 \rangle = \mathcal{F}_1 + \mathcal{F}_2, \quad (77a)$$

$$\mathcal{F}_1 = \frac{1}{2\pi} \int_0^\Delta dr \sqrt{4R^2 - r^2} \Omega^2(r), \quad (77b)$$

$$\mathcal{F}_2 = \frac{1}{2\pi} \int_\Delta^{2R} dr \sqrt{4R^2 - r^2} \Omega^2(r), \quad (77c)$$

where  $\Omega$  is given in Eq. (8), and  $1 \ll a\Delta \ll aR$ . For  $\mathcal{F}_1$  we have  $\mathcal{F}_1 \approx \frac{R}{2} \int_0^\Delta dr \Omega^2(r)$ , and since this is subdominant, depending only linearly on  $R$ , we discard it without further discussion. Inserting the asymptotic form  $\Omega^2(r) \approx 1/r^{2\beta}$  into  $\mathcal{F}_2$ , and writing  $s = r/(2R)$ , we have

$$\mathcal{F}_2 = \frac{R^{2-2\beta}}{2^{2\beta-1}\pi} \int_{\Delta/(2R)}^1 ds \sqrt{1 - s^2} / s^{2\beta}. \quad (78)$$

Passing to the limit  $\Delta/(2R) \rightarrow 0$  we obtain Eq. (76).

### 3. Physically realizable fields

How small can  $\beta$  be in practice? The answer is that asymptotically, i.e. for sufficiently large  $R$ ,  $\beta \geq 3/2$ . This is demonstrated below using the Van Cittert-Zernike theorem [21] and the fact that all physically realizable sources  $S(u)$  must be: (i) nonsingular, a requirement that excludes, inter alia, delta functions; (ii) positive definite; and (iii) strictly bounded, i.e.  $S(u > u_{max}) = 0$ .

In physically realizable sources whose finite extent is defined by a mask where the intensity falls discontinuously from some non-negligible value to zero at the mask edge  $u = u_{max}$ , we have for large  $r$

$$W(r) \approx \int^{u_{max}} u S(u) J_0(ru) du \approx -\sqrt{\frac{2}{\pi}} (u_{max})^{1/2} S(u_{max}) \cos(ru_{max} + \pi/4) / r^{3/2}, \quad (79)$$

so that both  $W(r)$  and  $W'(r)$  decay as  $r^{-3/2}$ , i.e.  $\beta = 3/2$ .

If  $S(u_{max})$  is extremely small the  $r^{-3/2}$  decay of  $W'$  sets in at values of  $r$  that may be so large as to be unmeasurable in practice. In that case, in practice one observes  $\beta \geq 4$ . To show this we set  $u_{max}$  equal to infinity and write

$$W(r) = \mathcal{W}_1 + \mathcal{W}_2, \quad (80a)$$

$$\mathcal{W}_1 = \int_0^\Delta u S(u) J_0(ru) du, \quad (80b)$$

$$\mathcal{W}_2 = \int_\Delta^\infty u S(u) J_0(ru) du, \quad (80c)$$

where  $1 \gg \Delta \gg 1/r$ . In  $\mathcal{W}_1$  we expand  $S(u) \approx S(0) + uS'(0)$  and obtain for large  $r$

$$\begin{aligned} \mathcal{W}_1 \approx & S(0) \left[ -\sqrt{\frac{2}{\pi}} \frac{\Delta^{1/2} \cos(r\Delta + \pi/4)}{r^{3/2}} + \frac{3}{8} \sqrt{\frac{2}{\pi}} \frac{\sin(r\Delta + \pi/4)}{\Delta^{1/2} r^{5/2}} \right] \\ & + S'(0) \left[ -\sqrt{\frac{2}{\pi}} \frac{\Delta^{3/2} \cos(r\Delta + \pi/4)}{r^{3/2}} + \frac{11}{8} \sqrt{\frac{2}{\pi}} \frac{\sin(r\Delta + \pi/4)}{r^{5/2}} - \frac{1}{r^3} \right]. \end{aligned} \quad (81)$$

Using the large argument expansion of  $J_0$  in  $\mathcal{W}_2$  and integrating by parts we have

$$\begin{aligned} \mathcal{W}_2 \approx & \sqrt{\frac{2}{\pi}} \int_\Delta^\infty u S(u) \left[ \frac{\sin(ru + \pi/4)}{\sqrt{ru}} - \frac{1}{8} \frac{\cos(ru + \pi/4)}{(ru)^{3/2}} \right] du \\ \approx & \sqrt{\frac{2}{\pi}} \frac{\cos(r\Delta + \pi/4)}{r^{3/2}} [S(0) + \Delta S'(0)] \Delta^{1/2} \end{aligned} \quad (82a)$$

$$\begin{aligned} & - \frac{\sin(r\Delta + \pi/4)}{r^{5/2}} \left( \frac{S(0) + \Delta S'(0)}{2\Delta^{1/2}} + S'(0)\Delta^{1/2} \right) \\ & + \frac{1}{8} \frac{\sin(r\Delta + \pi/4)}{r^{5/2}} [S(0) + \Delta S'(0)] \Delta^{-1/2} \end{aligned} \quad (82b)$$

$$\begin{aligned} = & S(0) \sqrt{\frac{2}{\pi}} \left[ \cos(r\Delta + \pi/4) r^{-3/2} \Delta^{1/2} + \frac{3}{8} \sin(r\Delta + \pi/4) r^{-5/2} \Delta^{-1/2} \right] \\ & + S'(0) \sqrt{\frac{2}{\pi}} \left[ \cos(r\Delta + \pi/4) r^{-3/2} \Delta^{3/2} - \frac{11}{8} \sin(r\Delta + \pi/4) r^{-5/2} \Delta^{1/2} \right] \end{aligned} \quad (82c)$$

Summing  $\mathcal{W}_1$  and  $\mathcal{W}_2$  yields

$$W(r) \approx -S'(0)/r^3, \quad (83)$$

i.e.  $\beta = 4$ . If, in addition,  $S'(0) = 0$ , the decay of  $W(r)$  is faster yet. For example, when all odd derivatives of  $S(u)$  vanish at the origin, as happens for a Gaussian  $S(u)$ ,  $W(r)$  decays exponentially fast.

Thus, for all physically realizable optical fields  $\langle Q^2 \rangle$  is bounded at its endpoints by two universal laws: for small  $R$ ,  $\langle Q^2 \rangle \sim R^2$ ; for large  $R$ ,  $\langle Q^2 \rangle \sim R$ . This behavior is illustrated in Fig. 5 for the practical  $J_0$  autocorrelation function in Section IV.



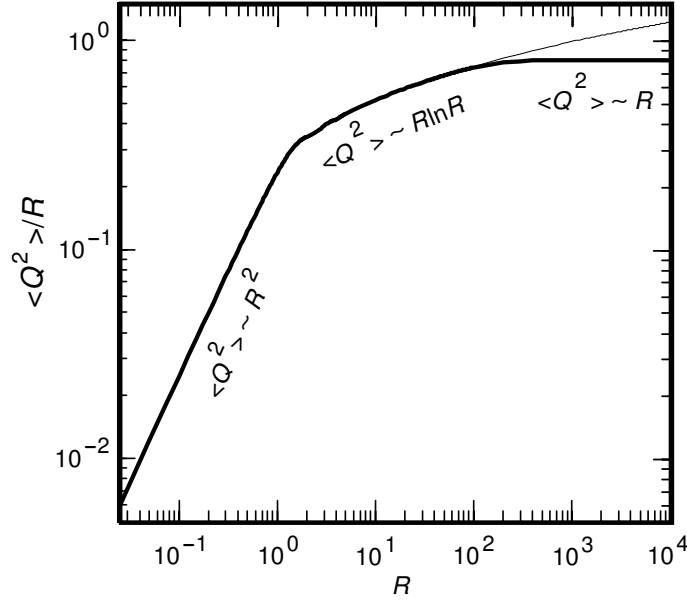


FIG. 5: Endpoint laws. Thick curve, practical  $J_0$ , Eq. (74a) with  $a = 1, b = 0.01$ ; thin curve, exact  $J_0$ , Eq. (35) with  $a = 1$ . Both functions obey the universal small  $R$  law,  $\langle Q^2 \rangle \sim R^2$ , Eq. (75), as they must, but, as expected, only the physically realizable practical  $J_0$  obeys the large  $R$  law,  $\langle Q^2 \rangle \sim R$ , Eq. (12).

## VI. SUMMARY

Under the assumption of circular Gaussian statistics, the topological charge variance  $\langle Q^2 \rangle$  of vortices in scalar fields [1] and C points in vector fields [2] was analyzed for a circular region of area  $A$  and radius  $R$  for both short and long range correlation functions.

(i) It was shown, Eq. (10), that when the autocorrelation function  $W(r)$  together with the first derivative  $W'(r)$  go to zero at infinity, Eq. (1) holds and there is screening.

(ii) For short range correlations  $\langle Q^2 \rangle$  grows linearly with  $R$ , Eq. (12): due to screening, inside  $A$  there is local charge neutralization, and only partially screened charges near the boundary contribute to the fluctuations. Very recent experimental measurements of  $\langle Q^2 \rangle$  [18] have used Eq. (12) to obtain the screening length  $\Lambda_s$ , a fundamental wavefield parameter. Comparison of  $\Lambda_s$  with the mean spacing between charges led to the conclusion that the charges form small clusters - a conclusion verified by direct imaging of the wavefield, Fig. 1.

(iii) Boundary smoothing [10] was considered for short range screening, and it was shown that such smoothing does not yield useful results, Eq. (32).

(iv) For long range correlation functions  $\langle Q^2 \rangle$  grows faster than  $R$ , and it is not possible to define a screening length. For a  $J_0$  correlation function  $\langle Q^2 \rangle \sim R \ln R$ , Eq. (46a). For correlation functions that decay more slowly than  $J_0$ ,  $\langle Q^2 \rangle \sim R^p$ , where  $1 < p < 2$ .

(v) Although a  $J_0$  correlation function is not attainable in practice, it was shown that one can generate an excellent approximation valid over an arbitrarily large, but finite range of  $R$ , Eq. (74a); this approximation was shown to yield results for  $\langle Q^2 \rangle$  that are in close agreement with those for  $J_0$ , Fig. 4.

(vi)  $\langle Q^2 \rangle$  can be calculated using either the charge correlation function  $C(r)$ , Eqs. (7) and (8), or the zero crossing correlation function  $\Gamma(r)$ , Eq. (53). An exact calculation showed that, as expected, these two seemingly different methods yield the same result for both short and long range screening.

(vii) For short range screening it was shown that it is also possible to obtain  $\langle Q^2 \rangle$  from zero crossing measurements made along any straight line of length  $P = 2\pi R$ , rather than only along the circular perimeter of  $A$ . For long range screening, however, it was shown that this useful simplification no longer holds.

(viii) It was also shown that for every physically realizable wavefield,  $\langle Q^2 \rangle \sim R^2$  for sufficiently small  $R$ , and that for sufficiently large  $R$ ,  $\langle Q^2 \rangle \sim R$ .

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